Conformal Klein-Gordon Equations and Quasinormal Modes

R. da Rocha[1,2,4] and E. Capelas de Oliveira[3]

Using conformal coordinates associated with conformal relativity—associated with de Sitter spacetime homeomorphic projection into Minkowski spacetime—we obtain a conformal Klein-Gordon partial differential equation, which is intimately related to the production of quasi-normal modes (QNMs) oscillations, in the context of electromagnetic and/or gravitational perturbations around, e.g., black holes. While QNMs arise as the solution of a wave-like equation with a Pöschl-Teller potential, here we deduce and analytically solve a conformal 'radial' d'Alembert-like equation, from which we *derive* QNMs formal solutions, in a proposed alternative to more completely describe QNMs. As a by-product we show that this 'radial' equation can be identified with a Schrödinger-like equation in which the potential is exactly the second Pöschl-Teller potential, and it can shed some new light on the investigations concerning QNMs.

1. INTRODUCTION

Quasi-normal modes (QNM) arise in the context of general relativity as electromagnetic or gravitational perturbations occurring in the neighborhood of, e.g., Schwarzschild, Kerr, Reissner-Nordstrøm (Jing, 2005), and Kerr-Newman spacetimes, and their investigation in the Schwarzschild background was started in Cunningham *et al*. (1979); Zerilli (1970, 1974). It is well known that no normal mode oscillations is produced in the process of emission of gravitational waves, but only quasi-normal oscillation modes, representing an oscillatory damped wave (Cunningham *et al*., 1979; Gleiser *et al*., 1996; Kokkotas, 1999; Leaver,

[1] DRCC—Instituto de Física Gleb Wataghin, Universidade Estadual de Campinas CP 6165, 13083-970 Campinas, SP, Brazil.

[2] Instituto de Física Teórica, Universidade Estadual Paulista Rua Pamplona 145, 01405-900 São Paulo, SP, Brazil.

[3] Departamento de Matemática Aplicada, IMECC, Unicamp, CP6065, 13083-859, Campinas, SP, Brazil.

[4] To whom correspondence should be addressed; e-mail: roldao@ifi.unicamp.br.

1980; Nollert, 1993; Press, 1971; Regge and Wheeler, 1957) while, as yet nothing is known for nonlinear stellar oscillations in general relativity (Detweiler and Ipser, 1973) and in the collapse of a star to form a black hole (Cunningham *et al.*, 1979). Also, a simple derivation of the imaginary parts of the QNM frequencies for the Schwarzschild-de Sitter spacetime is obtained in calculating the scattering amplitude in the first Born approximation and determining its poles (Padmanabhan, 2004; Roy Choudhury and Padmanabhan, 2004). As defined in, e.g. Khriplovich (2005); Kokkotas (1999), QNMs are the eigenmodes of the homogeneous wave equations, describing these perturbations – with the boundary conditions corresponding to outgoing waves at the spatial infinity and incoming waves at the horizon. The paramount interests to QNMs have been mainly introduced by Press (1971); Vishveshwara (1970). QNMs can bring imprints of black holes and can be detected in the gravitational wave framework (Kokkotas, 1999).

In the one hand, QNMs equations can be derived if we introduce a projective approach, namely the theory of hyperspherical universes, developed by Arcidiacono (2000) several years ago and, more specifically, in the so-called conformal case. When we write Maxwell equations in six dimensions, with six projective coordinates (we have, in these coordinates, a Pythagorean metric) a natural problem arises, namely, to provide a physical version of the formalism, i.e. to ascribe a physical meaning to the coordinates. For this theory, there are two possible different physical interpretations: a bitemporal interpretation and a biprojective interpretation. In the first case (bitemporal) we introduce a new universal constant c' and the coordinate $x_5 = i c' t'$ where t' is interpreted as a second time; we thus obtain in cosmic scale the so-called multitemporal relativity, proposed by Kalitzen (1975). The set of Maxwell equations obtained in this theory generalizes the equations of the unitary theory of electromagnetism and gravitation, as proposed by Corben (1946).

On the other hand (our second, biprojective case) we can interpret the extra coordinate, x_5, as a second projective coordinate. We then obtain the so-called conformal projective relativity, proposed by Arcidiacono (1985), which extends in cosmic scale the theory proposed by Ingraham (1954), but with a different physical interpretation. In this theory we have another universal constant, r_0, which can be taken as $r/r_0 = N$, where r is the radius of the hypersphere and N is the cosmological number appearing in the Eddington-Dirac theory (Eddington, 1920).

Here we consider only the second alternative, i.e., the biprojective interpretation. With this aim we introduce a projective space P_5 tangent to the hypersphere S^4. We then introduce six projective coordinates $\overline{x}_a$, with $a = 0, 1, \ldots, 5$ and normalized as

$$\overline{x}^2 + \overline{x}_0^2 - \overline{x}_5^2 = r^2,$$

where $\bar{x}^2 = x_i x^i$, $i = 1, \ldots, 4$ and r is the radius of the hypersphere. These coordinates allow us to construct the conformal projective relativity, using a six-dimensional tensor formalism.

This paper is organized as follows: in Section 2 we present a review of the so-called theory of hyperspherical universes, proposed by Arcidiacono, considering only the six-dimensional case, in which conformal projective relativity appears as a particular case. The choice of convenient coordinates and the link between the derivatives in these two formulations (a geometric version, six-dimensional, and a physical version, the five-dimensional conformal version) are also presented. After this review, we discuss in Section 3 a Klein-Gordon partial differential equation written in conformal coordinates. In Section 4, we show that a conformal 'radial' d'Alembert-like equation, can be led into a Schrödinger differential equation in which the associated potential is exactly a second Pöschl-Teller potential.

2. HYPERSPHERICAL UNIVERSES

In 1952 Fantappié proposed the so-called theory of the hyperspherical universes[4]. This theory is based on group theory and on the hypothesis that the universe is endowed with unique physical laws, valid for all observers. As particular cases, Arcidiacono (2000) studied a limitation of that theory, i.e., he considered hyperspherical universes with $3, 4, \ldots, n$ dimensions where motions are given by $n(n + 1)/2$-parameter rotation group in spaces with $4, 5, \ldots, (n + 1)$ dimensions, respectively. Those models of hyperspherical $S^3, S^4, \ldots, S^n$ universes, can be interpreted as successive physical improvements, because any one of them (after S^4) contains its precedents and is contained in its successors.

After 1955 Arcidiacono studied the case $n = 4$, special projective relativity, based on the de Sitter hyperspherical universe with a group (the so-called Fantappié-de Sitter group) of ten parameters. This theory is an improvement (in a unique way) of Einstein's special relativity theory and provides a new group-theoretical version of the big-bang cosmology. As a by-product of special projective relativity one can recover several results, for example, Kinematic Relativity, proposed by Milne (1935, 1948); Stationary Cosmology, proposed by Bondi and Gold (1948) and Plasma Cosmology, proposed by Alfvèn (1983, 1986).

Moreover, if we consider a universe S^4 as globally hyperspherical but endowed with a locally variable curvature, we obtain the so-called general projective relativity which was proposed and studied by Arcidiacono after 1964. This theory allows us to recover several results as particular cases, for example, the unitary theories proposed by Jordan (1947), Kaluza (1921); Klein (1926, 1928), Straneo (1931), Thiry (1951), Veblen (1933) and Weyl (1918a,b, 1919, 1952) and some

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[4] See the Appendix A.

generalizations of the gravitational field, as those proposed by Brans and Dicke (1961), Sciama (1958) and Rosen (1973, 1980).

In this paper we are interested only in the case $n = 5$, i.e., conformal projective relativity based on the hyperspherical universe S^4 and its associated rotation group, with fifteen parameters, which contains the accelerated motions. We remember that, whereas for $n = 4$ we have a unitary theory (a magnetohydrodynamic field), for $n = 5$ we have another unitary theory, i.e., the magnetohydrodynamics and Newton's gravitation. We also present the relations between Cartesian, projective and conformal coordinates and the link involving derivatives in the six- and five-dimensional formulations.

2.1. Conformal Coordinates

We use the notation x_i, $(i = 1, 2, 3, 4)$ and x_5 for conformal coordinates and $\overline{x}_a$, $(a = 0, 1, 2, 3, 4, 5)$ for projective coordinates. The relations between these coordinates are

$$x_i = r_0 \frac{\overline{x}_i}{\overline{x}_0 + \overline{x}_5} \qquad \text{and} \qquad x_5 = r_0 \frac{r}{\overline{x}_0 + \overline{x}_5},$$

which satisfy the condition

$$x_5^2 - x^2 = r_0^2 \frac{\overline{x}_0 - \overline{x}_5}{\overline{x}_0 + \overline{x}_5},$$

where $x^2 = x_i x^i$, and r_0 and r are constants. After these considerations, the transformations of the so-called conformal projective group are obtained using the quadratic form in projective coordinates

$$\overline{x}^2 + \overline{x}_0^2 - \overline{x}_5^2 = r^2,$$

decomposing the elements of the six-dimensional rotation group (with fifteen parameters) in fifteen simple rotations $(\overline{x}_a, \overline{x}_b)$.

2.2. Connection Between Derivatives

Our main objective is to write down a differential equation, more precisely a Klein-Gordon-like equation, associated with conformal coordinates. We first obtain the relation between the six projective derivatives $\overline{\partial}_a \equiv \partial/\partial \overline{x}_a$ and the five-dimensional derivatives $\partial_i = \partial/\partial x_i$ and $\partial_5 = \partial/\partial x_5$. We can then write the differential equations in the projective formalism, with six dimensions, in physical, i.e., conformal coordinates, with five dimensions.[5]

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[5] As we already know, in five dimensions we must impose a condition on space in order to account for the fact that we are aware of only four dimensions. We have the same situation here, i.e., we must impose an additional condition.

Taking $\phi = \phi(x_i, x_5)$, a scalar field, and using the chain rule we can write

$$\partial_i \phi = \left[(\bar\partial_i \bar x_k)\bar\partial_k + (\bar\partial_i \bar x_5)\bar\partial_5 + (\bar\partial_i \bar x_0)\bar\partial_0\right]\bar\phi$$

$$\partial_5 \phi = \left[(\bar\partial_5 \bar x_k)\bar\partial_k + (\bar\partial_5 \bar x_5)\bar\partial_5 + (\bar\partial_5 \bar x_0)\bar\partial_0\right]\bar\phi$$

with $\bar\phi = \bar\phi(\bar x_i, \bar x_5, \bar x_0)$ and $i, k = 1, 2, 3, 4$.

From now on we take $r = 1 = r_0$. We consider $\bar\phi(\bar x_a)$ a homogeneous function with degree N in all six projective coordinates $\bar x_a$. Using Euler's theorem associated with homogeneous function, we get

$$\left(\bar x_i \bar\partial_i + \bar x_5 \bar\partial_5 + \bar x_0 \bar\partial_0\right)\phi = N\phi$$

where $\bar\partial_a = \partial/\partial\bar x_a$ and N is the degree of homogeneity of the function.

Then, the link between the derivatives can be written as follows (Arcidiacono and Capelas de Oliveira, 1996)

$$\bar\partial_0\,\bar\phi = N\frac{A^+}{x_5}\phi + B^-\partial_5\phi - x_5 x_i \partial_i\phi$$

$$\bar\partial_5\,\bar\phi = -N\frac{A^-}{x_5}\phi - B^+\partial_5\phi - x_5 x_i \partial_i\phi$$

$$\bar\partial_i\,\bar\phi = N\frac{x_i}{x_5}\phi + x_i \partial_5\phi + x_5 \partial_i\phi$$

where we have introduced a convenient notation

$$2A^\pm = 1 \mp x^2 \pm x_5^2 \qquad \text{and} \qquad 2B^\pm = 1 \pm x^2 \pm x_5^2.$$

We observe that for $\bar x_5 = 0$ and considering $\bar\partial_5\bar\phi = 0$ we obtain

$$\bar\partial_i\bar\phi = A\partial_i\phi + \frac{N}{A}x_i\phi$$

$$\bar\partial_0\bar\phi = -Ax_i\partial_i\phi + \frac{N}{A}\phi$$

where $A^2 = 1 + x^2$. These expressions are the same expressions obtained in special projective relativity (Gomes and Capelas de Oliveira, 2004) and provide the link between the five projective derivatives and the four derivatives in Cartesian coordinates, i.e., the relation between five-dimensional (de Sitter) universe and four-dimensional (Minkowski) universe.

3. CONFORMAL KLEIN-GORDON EQUATION

In this section we use the previous results to calculate the so-called generalized Klein-Gordon differential equation

$$\frac{\partial^2}{\partial \overline{x}_a^2} \Phi + m^2 \Phi = 0$$

where m^2 is a constant and $a = 0, 1, \ldots, 5$. Introducing projective coordinates (in this case we have a Pythagorean metric) we obtain[6]

$$\frac{\partial^2 U}{\partial \overline{x}_i^2} + \frac{\partial^2 U}{\partial \overline{x}_0^2} - \frac{\partial^2 U}{\partial \overline{x}_5^2} + m^2 U = 0$$

where $i = 1, 2, 3, 4$ and $U = U(\overline{x}_i, \overline{x}_0, \overline{x}_5)$.

Using the relations between projective and conformal coordinates and the link (involving the derivatives) in the two formulations we can write

$$\left[x_5^2 \left(\Box - \frac{\partial^2}{\partial x_5^2} \right) + 3 x_5 \frac{\partial}{\partial x_5} + N(N + 5) + m^2 \right] u(x_i, x_5) = 0$$

where N and m^2 are constants, $\Box$ is the Dalembertian operator given by

$$\Box = \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$$

and Δ is the Laplacian operator. This partial differential equation is the so-called Klein-Gordon differential equation written in conformal coordinates or a conformal Klein-Gordon equation.

The case $m^2 = 0$ transforms this equation in the so-called generalized d'Alembert differential equation. Another way to obtain this differential equation is to consider the conformal metric in Cartesian coordinates, which furnishes the so-called Beltrami metric (Arcidiacono, 2000) where the d'Alembert equation appears naturally. This equation can also be obtained by means of the second order Casimir invariant operator[7] associated with the conformal group.

To solve the conformal Klein-Gordon equation, we first introduce the spherical coordinates (r, θ, ϕ) and get

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \mathcal{L}u - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_5^2} + \frac{3}{x_5} \frac{\partial u}{\partial x_5} + \frac{\Lambda}{x_5^2} u = 0, \qquad (1)$$

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[6] Hereafter we consider $m = m_0 c / \hbar$ where m_0, c and $\hbar$ have the usual meanings.
[7] Invariant operators associated with dynamic groups furnish mass formulas, energy spectra and, in general, characterize specific properties of physical systems.

where we introduced $x_4 = ict$ and defined the operator[8]

$$\mathcal{L} \equiv \frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \tag{2}$$

involving only the angular part. In this partial differential equation we have $u = u(r, \theta, \phi, t, x_5)$ with $\Lambda = N(N+5) + m^2$.

Using the method of separation of variables we can eliminate the temporal and angular parts, writing

$$u = u(r, \theta, \phi, t, x_5) = A \, e^{inct} Y_{\ell m}(\theta, \phi) \, f(r, x_5), \tag{3}$$

where A is an arbitrary constant, $n > 0$, $\ell = 0, 1, \ldots$ and $m = 0, \pm 1, \ldots$ with $-\ell \le m \le \ell$ and $Y_{\ell m}(\theta, \phi)$ are the spherical harmonics, we get the following partial differential equation

$$\frac{\partial^2 f}{\partial r^2} + \frac{2}{r} \frac{\partial f}{\partial r} - \frac{\partial^2 f}{\partial x_5^2} + \frac{3}{x_5} \frac{\partial f}{\partial x_5} + \frac{\Lambda}{x_5^2} f + \left[n^2 - \frac{\ell(\ell+1)}{r^2} \right] f = 0$$

with $f = f(r, x_5)$. If we impose a regular solution at the origin ($r \to 0$), the solution of this partial differential equation can be obtained in terms of a product of two Bessel functions.

4. A D'ALEMBERT-LIKE EQUATION

In this section we present and discuss a partial differential equation which can be identified to a d'Alembert-like equation, which we call a conformal 'radial' d'Alembert equation. We firstly introduce a convenient new set of coordinates, then we use separation of variables and obtain two ordinary differential equations. One of them can be identified as an ordinary differential equation whose solution is a generalization of Newton's law of gravitation; the other one is identified with an ordinary differential equation similar to a one-dimensional Schrödinger differential equation with a potential equal to the second Pöschl-Teller potential (da Rocha and Capelas de Oliveira (2005)).

We introduce the following change of independent variables

$$r = \rho \cosh\xi,$$

$$x_5 = \rho \sinh\xi, \tag{4}$$

with $\rho > 0$ and $\xi \ge 0$, in the separated Klein-Gordon equation, obtained in the previous section, and after another separation of variables we can write a pair of

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[8] Here r is a coordinate and should not be confused with the radius of the hypersphere. Besides, it is always possible to define a Wick-rotation (Sakurai, 1985) of the time coordinate, i.e., $ct \mapsto ict$.

ordinary differential equations, namely,

$$\rho^2 \frac{d^2 U}{d\rho^2} - p(p+1)U = 0, \tag{5}$$

where $U = U(\rho)$ and

$$\frac{d^2 V}{d\xi^2} + (2\tanh\xi - 3\coth\xi)\frac{dV}{d\xi} + \left[\frac{\ell(\ell+1)}{\cosh^2\xi} - \frac{\Lambda}{\sinh^2\xi} - p(p+1)\right]V = 0 \tag{6}$$

where $V = V(\xi)$ and p is a separation constant.

We first discuss Eq. (5). Its general solution is given by

$$U(\rho) = C_1 \rho^{-p} + C_2 \rho^{p+1}$$

where C_1 and C_2 are arbitrary constants.

If we consider the case $p = 1$, introducing the notation $C_1 = gM$ with g and M having the usual meanings, we get

$$U(\rho) = \frac{gM}{\rho} + C_2 \rho^2$$

i.e., a gravitational potential which can be interpreted as a sum of a Kepler-like potential and a harmonic oscillator potential, giving rise to the gravitational force

$$f(\rho) = \nabla U = -\frac{gM}{x^2 - x_5^2} + 2C_2(x^2 - x_5^2)^{1/2},$$

with a singularity at $x = x_5$. We note that for $C_2 = 0$ we obtain an expression analogous to Newton's law of gravitation.

Secondly, Eq. (6) can be solved by introducing the change of dependent variable

$$V(\xi) = \sinh^{\frac{1}{2}}\xi \tan h\,\xi\, F(\xi)$$

and we obtain

$$-\frac{d^2}{d\xi^2} F(\xi) + \left[\frac{\mu(\mu-1)}{\sinh^2\xi} - \frac{\ell(\ell+1)}{\cosh^2\xi} + \left(p + \frac{1}{2}\right)^2\right] F(\xi) = 0, \tag{7}$$

where the parameter μ is given by a root of the algebraic equation $\mu(\mu-1) = N^2 + 5N + 15/4 + m^2$.

The differential equation above can be identified with a Schrödinger-like differential equation in which the associated potential is given by

$$\mathcal{V}_{\mu\ell}(\xi) = \frac{\mu(\mu-1)}{\sinh^2\xi} - \frac{\ell(\ell+1)}{\cosh^2\xi}, \tag{8}$$
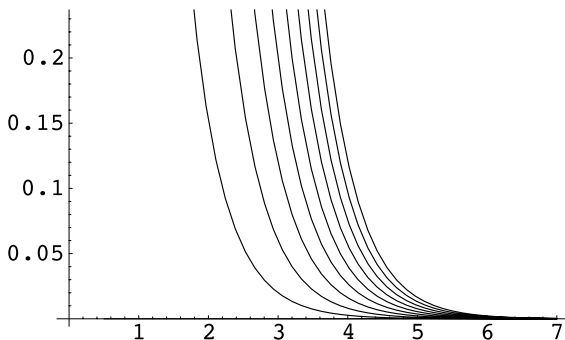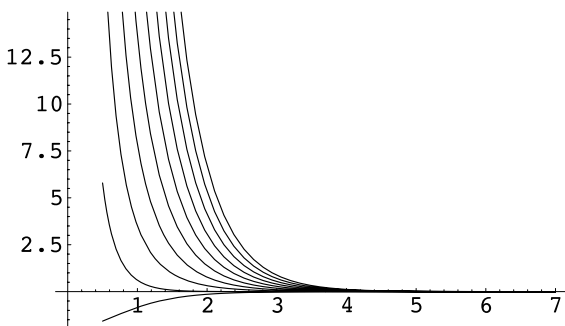
Fig. 1. $V_{\mu\ell}(\xi) \times \xi$ evaluated for $\mu = \{1, \ldots, 10\}$ and $\ell = 0, 1$.

which is exactly the second Pöschl-Teller potential with energy E given by $E_p = -(p + 1/2)^2 < 0$. The graphics showing $V_{\mu\ell}(\xi)$ for various values of μ and ℓ are depicted below (Figs. 1–10).

Explicit solutions of Eq. (7) are written as

$$F(\xi) = (\tanh^{-1/2} \xi)(\tanh^2 \xi - 1)^\sigma \times$$

$$\times \left({}_2F_1(\sigma_1, \sigma_2, \sigma_3, \tanh^2 \xi)(\tanh^{-\sigma_3} \xi)D \right.$$

$$\left. + {}_2F_1(1 + \sigma_3 + \sigma_1, 1 + \sigma_2 + \sigma_3, 2 + \sigma_3, \tanh^2 \xi)(\tanh^{2+\sigma_3} \xi)E \right)$$

where D, E are integration constants, ${}_2F_1$ denotes the hypergeometric function, and

$$\sigma_1 = \frac{1}{4}(3 - 2\mu - 2\ell - 2p), \quad \sigma_2 = \frac{1}{4}(1 - 2\mu + 2\ell - 2p),$$

$$\sigma_3 = \frac{1}{2}(2\mu - 3), \quad 2\sigma = \sigma_1 + \sigma_2 + \sigma_3 \tag{9}$$
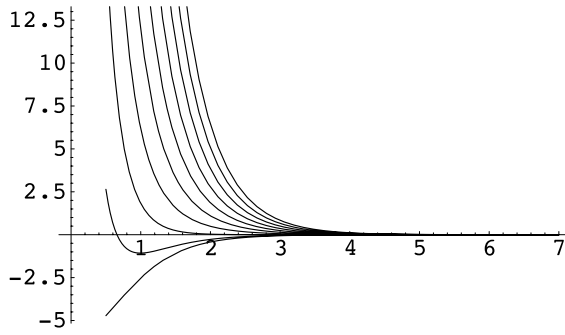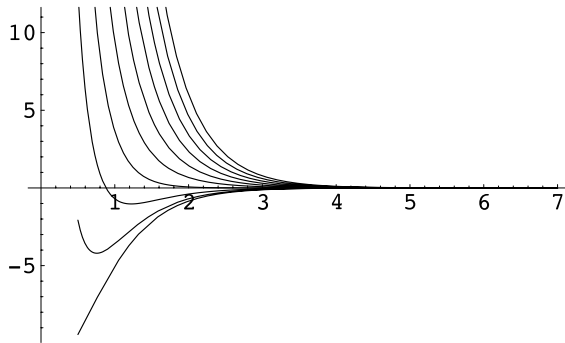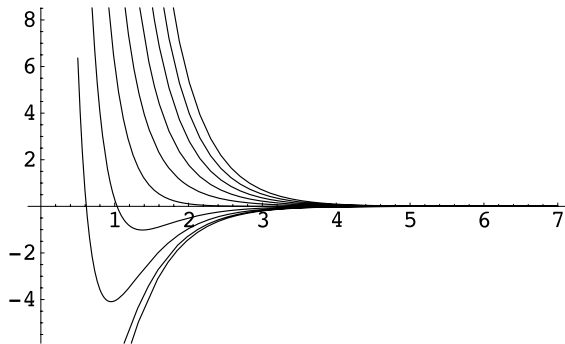

Fig. 2. $V_{\mu\ell}(\xi) \times \xi$ evaluated for $\mu = \{1, \ldots, 10\}$ and $\ell = 2$.
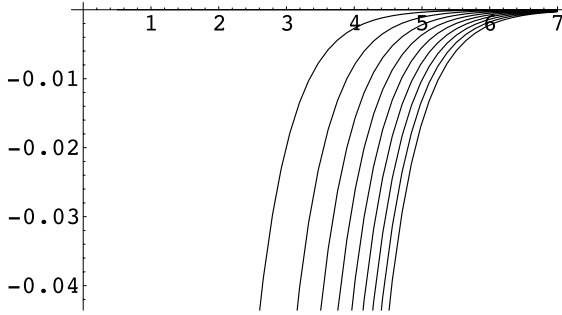
Fig. 3. $V_{\mu\ell}(\xi) \times \xi$ evaluated for $\mu = \{1, \ldots, 10\}$ and $\ell = 3$.
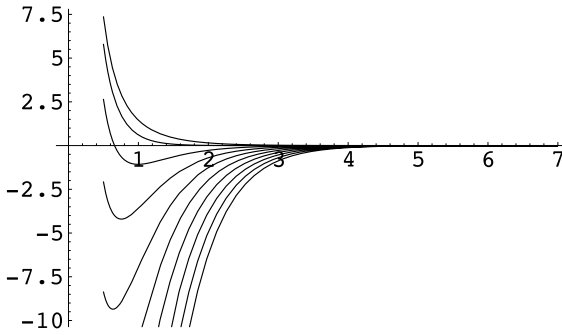

Fig. 4. $V_{\mu\ell}(\xi) \times \xi$ evaluated for $\mu = \{1, \ldots, 10\}$ and $\ell = 4$.
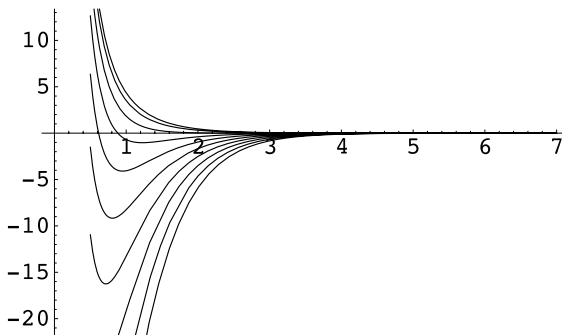

Fig. 5. $V_{\mu\ell}(\xi) \times \xi$ evaluated for $\mu = \{1, \ldots, 10\}$ and $\ell = 5$.

Fig. 6. $V_{\mu\ell}(\xi) \times \xi$ evaluated for $\ell = \{1, \ldots, 10\}$ and $\mu = 0, 1$.

Fig. 7. $V_{\mu\ell}(\xi) \times \xi$ evaluated for $\ell = \{1, \ldots, 10\}$ and $\mu = 2$.

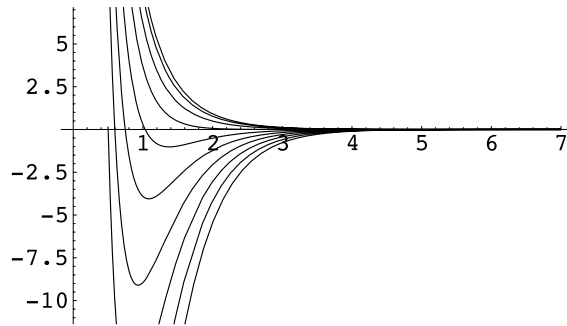**Fig. 8.** $V_{\mu\ell}(\xi) \times \xi$ evaluated for $\ell = \{1, \ldots, 10\}$ and $\mu = 3$.
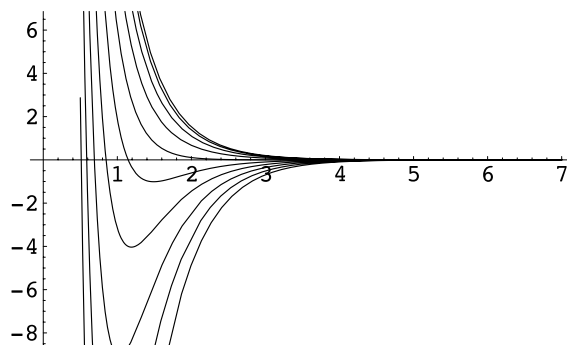
Fig. 9. $V_{\mu\ell}(\xi) \times \xi$ evaluated for $\ell = \{1, \ldots, 10\}$ and $\mu = 4$.

We note that the first Pöschl-Teller potential is connected to the study of a Dirac particle on central backgrounds associated with an anti-de Sitter oscillator, i.e., the transformed radial wave functions satisfy the second-order Schrödinger differential equation whose potential is exactly the first Pöschl-Teller potential (Cotăescu, 1998).

5. QNMS EQUATIONS

It is also possible to derive QNMs formal partial differential equations if, instead the *ansatz* given by Eq. (3), we assume

$$u = u(r, \theta, \phi, t, x_5) = e^{\pm im\phi} \, F(r, \theta, t, x_5), \tag{10}$$

Fig. 10. $V_{\mu\ell}(\xi) \times \xi$ evaluated for $\ell = \{1, \ldots, 10\}$ and $\mu = 5$.

and Eq. (1) reads

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{m^2}{r^2 \sin^2 \theta} - \right.$$

$$\left. \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_5^2} + \frac{3}{x_5} \frac{\partial}{\partial x_5} + \frac{\Lambda}{x_5^2} \right) F(r, \theta, t, x_5) = 0, \qquad (11)$$

which, after the separation of variables given by $F(r, \theta, t, x_5) = \Theta(\theta) R(r, t, x_5)$, can be lead to

$$\Theta''(\theta) + \cot \theta \, \Theta'(\theta) + [\ell(\ell + 1) - m^2 \csc^2 \theta] \Theta(\theta) = 0, \qquad (12)$$

which presents the associated Legendre polynomials $\Theta(\theta) = P_\ell^m(\theta)$ as solutions[9], and

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_5^2} + \frac{3}{x_5} \frac{\partial}{\partial x_5} + \left(\frac{\Lambda}{x_5^2} - \frac{\ell(\ell + 1)}{r^2} \right) \right) R(r, t, x_5) = 0, \qquad (13)$$

Now if we use again Eqs. (4), then Eq. (13) takes the form

$$- \frac{\partial^2 R}{\partial t^2} + \frac{\partial^2 R}{\partial \rho^2} - \frac{1}{\rho^2} \left[\frac{\partial^2}{\partial \xi^2} + (2 \tanh \xi - 3 \coth \xi) \frac{\partial}{\partial \xi} \right.$$

$$\left. + \left(\frac{\ell(\ell + 1)}{\cosh^2 \xi} - \frac{\Lambda}{\sinh^2 \xi} \right) \right] R(\xi, t, \rho) = 0 \qquad (14)$$

where now $R = R(\xi, t, x_5)$. By means of the *ansatz*

$$R(\xi, t, x_5) = \frac{\sinh^{3/2}(\xi)}{\cosh(\xi)} H(\xi, t, x_5), \qquad (15)$$

and making $\rho = 1$, which implies by Eqs. (4) that $r^2 - x_5^2 = 1$, Eq. (14) gives

$$\frac{\partial^2}{\partial t^2} + \left[\frac{\partial^2}{\partial \xi^2} - V_{\mu, \ell}(\xi) - \frac{1}{4} \right] u_1(\xi, t) = 0, \qquad (16)$$

where $u_1(\xi, t) = R(\xi, t, \rho = 1)$. The potential $V_{\mu, \ell}(\xi)$ is defined by Eq. (7), where $\mu(\mu - 1) = \Lambda + 15/4$. These equations are formally QNMs equations and are prominently used in the description of gravitational and electromagnetic perturbations in supermassive stellar and black holes processes.

6. CONCLUDING REMARKS

In this paper we discussed the calculation of a conformal d'Alembert-like equation. We used the methodology of projective relativity to obtain a conformal Klein-Gordon differential equation and, after the separation of variables, we

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[9] Here we have imposed regularity condition, i.e., for $\theta = \pm\pi$, the solution is analytic.

got another partial differential equation in only two independent variables, the so-called conformal d'Alembert differential equation. Another separation of variables led to an ordinary differential equation which generalizes Newton's law of gravitation. Finally, we have shown that the remaining differential equation, a 'radial' differential equation, is transformed into a one-dimensional Schrödinger differential equation with an associated potential that can be identified exactly with the second Pöschl-Teller potential. These equations can shed some new light on the analytical investigations concerning QNMs. There are some open questions concerning QNM modes, and new developments are promising, and not only restricted to $d = 4$ spacetimes. Motl and Neitzke (2003) has recently given an original derivation of the quasinormal frequencies of Schwarzschild black holes in the case where $d \geq 4$, and also of Reissner-Nordstrøm black holes in $d = 4$, in the limit of infinite damping, where they find an aperiodic behavior for the QNM for spin 0,1, and 2 fields (Motl and Neitzke, 2003). Also, in Shao *et al.* (2005); Wang *et al.* (2000) QNMs are investigated in a time-dependent black hole background and in Reissner-Nordstrøm-AdS black holes.

From supersymmetric quantum mechanics with periodic potentials, it can be seen that the most general periodic potentials which can be analytically solved involve Jacobi's elliptic functions, which in various limits become Pöschl-Teller potentials arising in the context of Kaluza-Klein spectrum (Kaluza, 1921). Kaluza-Klein modes of the graviton have been widely investigated (Brandhuber and Sfetsos, 1999; Hatanaka *et al.*, 1999; Nam, 2000; Randall and Sundrum, 1999a,b), since the original formulation of Randall and Sundrum necessarily has a continuum of Kaluza-Klein modes without any mass gap, arising from a periodic system of 3-branes. The methods and equations developed here can shed some new light in the calculation of mass gaps from a distribution of D-branes (Nam, 2000) in the context of five-dimensional supergravity – which will be discussed in a forthcoming paper.

A natural continuation of this calculation is to prove that all 'radial' problems associated with an equation resulting from a problem involving a light cone can be led into a Schrödinger-like differential equation in which the potential is exactly the Pöschl-Teller potential (Capelas de Oliveira, 2005).

APPENDIX: HYPERSPHERICAL UNIVERSE MODELS

In this appendix we briefly summarize the idea of hyperspherical universes as originally proposed by Fantappié (1973) and developed by Arcidiacono (2000).

The main motivation behind those models is to consider seriously the premise that a Universe must be a harmonic and well ordered system of laws, and that this statement is to be expressed mathematically by using group theory in an appropriate way. Taking into account that Galilean Relativity, which uses the Galileo group as invariance group of physical laws, has been perfected into

Special Relativity, which uses the Poincaré group as invariance group of physical laws, Fantappié asked himself in which way it would be possible to perfect Special Relativity into a kind of *final* relativity. His answer to this question was very simple indeed. He realized that the Poincaré group is the contraction of the 10-parameter Lie group known today as de Sitter group (but which should be called, as in the text, the Fantappié-de Sitter group) which can be made to act projectively on a flat 4-dimensional (Minkowski) spacetime. Next Fantappié asked whether there were other spacetime manifolds where the group could act naturally and serve as invariance group of physical laws. The answer is positive, and Fantappié found that the natural manifold is a hyperspherical universe, which he called S^4, of constant curvature radius. Of course, the previously known models for the universe (not based on General Relativity) are all particular cases of this one, corresponding to some group contractions involving the velocity of light and/or the universe radius, or both. Fantappié did not stop there. He proposed that the hyperspherical universe model S^4 was only an approximation for truth in the sense that it was embedded in a hyperspherical universe model S^5 where the conformal group (a 15-parameter Lie group) acts naturally as invariance group of physical laws. By its turn, S^5 may be generalized into S^6 and so on. At each generalization new fundamental physical constants make their natural appearance as a kind of group parameter whose contraction produces the group used in the previous universe model. It is clear that the extra dimensions in each Universe model must be interpreted, in an appropriate way (something that is also necessary in modern Kaluza-Klein type theories), and it is at this point that his mathematical skills give us useful physical hints. In particular, Arcidiacono, one of his students, showed that the hyperspherical universe models S^4 and S^5 contain many aspects of several proposed unified theories. Of course, we do not have space here even to start discussing the many beautiful results found by Arcidiacono and we invite the reader to consult Arcidiacono book (Arcidiacono, 2000) for more details. However, we would like to emphasize here that many ideas proposed by him are worth to be more developed since, in particular, it seems that in his work there is the seed of a simple solution for the problem of dark energy and dark matter, an issue that we shall discuss elsewhere.

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